Stress fluctuations and shear zones in quasistatic granular chute flows

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Experimental investigations of quasistatic flows of a two-dimensional granular material comprised of circular cylindrical rods are described. The flows are confined by two rough, parallel walls that can be inclined to the vertical direction. It is observed that boundary layers develop next to the sidewalls and are approximately 5 particle diam. thick, regardless of the width of the channel. The flows are analyzed by means of a modification of Coulomb theory that takes into account stress fluctuations that arise as a result of the "granularity" of the flowing material. We propose a probabilistic model where the stress fluctuations play the role of temperature, and the difference between the shear stress and yield stress plays the role of an energy barrier that the system has to overcome in order to yield. The velocity profiles predicted by the model are in agreement with the experiments for both vertical and inclined bins.

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I. INTRODUCTION

Despite the wide range of applications involving particulate transport, the mechanics of granular flows is not well understood. The fascinating behavior of grains [1] is, however, the subject of more and more research. In granular flows, there exist two well-defined asymptotic regimes: rapid flows, where internal stresses are mainly due to the collisions between grains, and quasistatic flows, where the stresses are principally due to the friction between particles. For the time being, no complete theory which includes these different mechanisms exists. Rapid granular flows have been successfully described using kinetic theories [2,3], in which the concept of granular temperature was introduced in order to account for the velocity fluctuations. In the case of quasistatic flows, continuum soil mechanics is often used. Although the continuum approach seems to be appropriate for the prediction of deformation of materials under loads, paradoxes appear when one attempts to use it for fully developed quasistatic flows [4].

In this paper we study quasistatic flows of granular material in parallel sided bins. These are often encountered in industrial processes such as hopper flows or moving bed reactors. Because of its simplicity, the flow of particles between two parallel rough walls is also of great fundamental interest, allowing us to better understand the basic features of quasistatic granular flows.

In a vertical bin away from the top and the outlet, the flow exhibits a velocity profile characterized by a plug flow in the central region and shear zones near the walls (Fig. 1). Nedderman and Laohakul [5] have carried out systematic experiments to measure the size of the shear zones, checking the influence of the flow parameters. Using different kinds of particles, they found that the thickness of the shear zones scales with the particle size, being approximately equal to 10 particle diameters. The thickness was found to be independent of the flow rate imposed at the outlet, of the wall roughness, and of the width of the channel, except for very wide channels where a slight increase of the shear zone was observed.

In the literature one can find many other measurements of velocity profiles in vertical chute flows. In an axisymmetric configuration, Takahashi and Yanai [6] found a shear zone at the wall having a thickness of 5 particle diameters. Savage [7] obtained a boundary layer thickness of 10 particle diameters. Using fine particles, Savage [8] performed experiments in channels of different widths, and observed shear zones whose thicknesses were approximately 15 particle diameters and independent of the bin width. Natarajan, Hunt, and Taylor [9] reported velocity profiles in which the boundary layers extend over 6 particle diameters. In all these experiments, the

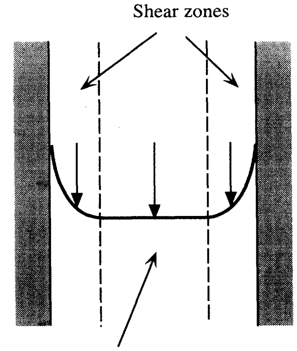

FIG. 1. Sketch of the velocity profile in a vertical bin.

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characteristic length scale that determines the development of the shear zone seems to be the size of the particles, and not the size of the bin.

Another problem where phenomena that scale on the particle size are observed is the problem of localization of failures in triaxial tests. It is well known that when a granular medium is constrained in a triaxial cell, the deformation is localized in shear bands whose thicknesses vary between 10 and 15 particle diameters. This localization of failures has been the subject of many experimental and numerical studies [10–13].

In both problems, i.e., the development of the shear zones in the vertical chute flow and the formation of failures in soils, the characteristic length scale appears to be the size of the particles. Thus any description using classical continuum plastic theory fails, as the granularity of the material is not taken into account. For the problem of the shear bands some attempts have been made to explain the thickness of the bands by introducing the rotation of the particles through a Cosserat theory [11]. This approach correctly reproduces the finite thickness of the failure but lacks solid experimental validation since the only direct measurement that can be made is the width of the shear band. For the development of the shear zones in vertical chute flows no satisfactory explanation exists, and no theory is able to correctly predict the velocity profiles experimentally observed [14].

In the present investigation, we perform experiments of quasistatic chute flows using a two-dimensional granular material comprised of circular cylindrical rods. We also propose a probabilistic model which is able to predict the formation of the shear zones. The paper is organized as follows. Section II describes the experimental setup. Velocity and concentration profiles in the vertical bins are presented in Sec. III. Both the velocity and the concentration exhibit boundary layers of constant thickness. In Sec. IV, we show how the formation of the shear zones can be interpreted in terms of the stress fluctuations that arise from the underlying discrete force network. The probabilistic model is presented in Sec. V. This analysis enables one to account for the fluctuations in a continuum Coulomb-type theory. We show that the velocity profiles predicted by this approach are in agreement with the experimental measurements, in both the vertical (Sec. V) and the inclined chute flows (Sec. VI). Finally, the main results of the study are summarized in Sec. VII.

In the following, starred and unstarred quantities denote dimensional and nondimensional variables, respectively.

II. EXPERIMENTAL SETUP AND PROCEDURE

Aluminum cylinders 6 cm long were used as two-dimensional granular material. In order to avoid the formation of a regular packing, the medium was made up of an equal mixture of 2 and 3 mm diameter cylinders (the mean particle diameter d^* is 2.5 mm). The density of the rods is 2.7 g/cm^3 and the aluminum-aluminum friction angle is equal to 19.5°.

The rods were contained in a vertical bin made of two parallel rough sidewalls 1.2 m long and 6 cm wide, with a vertically moving bottom plate (Fig. 2). The length of the cylinders being quite large compared to their diameter, no lateral motion took place and consequently no front or back plates were required to hold the rods. By using a two-dimensional granular material, we were able to get rid of spurious effects due to friction with surfaces other than the sidewalls. The lateral walls were roughened by gluing 2.5 mm diameter plastic rods to the surface. The distance between the two layers of plastic rods is $2b^*$ as shown in Fig. 2. This gap can vary from 2.5 to 11 cm, which allowed us to perform experiments in channels with different widths.

The internal friction angle ϕ of the granular material is defined as the maximum slope a pile can reach and was found to be 21° for a packing with a concentration of $C=0.89$. The wall friction angle δ between the aluminum cylinders and the sidewalls was found by slowly inclining the wall and measuring the critical angle at which the material began to slide. We found $\delta=21°$, the same value found for ϕ.

The position of the moving bottom that closes the channel is controlled by a positioning lift. The maximum displacement allowable in the setup is 20 cm. Before each experiment, the positioning lift is moved up to its maximum position and the channel is filled with rods. The large height of the apparatus compared to its width guarantees that a state of constant pressure is reached. The flow is created by slowly and carefully moving down the bottom plate by hand, at a velocity never exceeding 1 mm/s. Under such conditions, the flow belongs to a quasistatic regime, where collective inertial effects are negligible. As a dilatation is necessary for the material to move, a density wave that propagates from the bottom to the free surface is observed during the first steps of the displacement. Once the medium is sufficiently dilated, a steady flow is reached. An initial displacement of 5 cm of the bottom plate was sufficient to eliminate the transient.

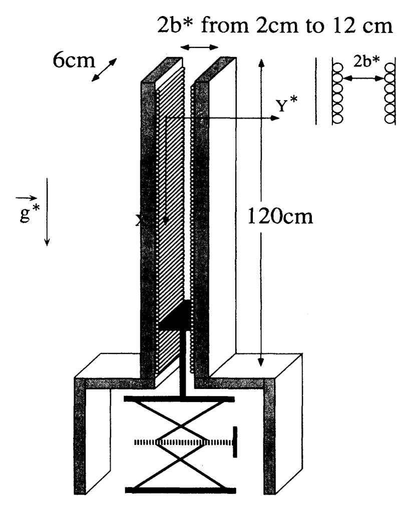

FIG. 2. Experimental setup.

The results presented here were obtained after a steady flow was reached. The measurements were carried out in the middle region of the bin, at a depth sufficient to get a constant pressure, and away from the bottom plate where the streamlines are not parallel to the walls (Fig. 3). In the region of measurements, the flow is uniform along the channel and only depends on the transversal coordinate y^*.

Two kinds of measurements were carried out: velocity profiles $u(y)$ and concentration profiles $C(y)$. In fact, we did not measured velocities, but relative displacements $\Delta x^*(y)/\Delta X^*$, where $\Delta x^*(y)$ is the distance covered by the particles when a displacement ΔX^* is imposed on the bottom plate. In the quasistatic regime, the velocity of the bottom plate V^* has no effect on the nondimensional velocity of the material $u(y)=u^*(y)/V^*$. One can then easily be convinced that in this regime the relative velocity and the relative displacement are the same. As long as $u(y)$ is independent of V^* one has

$$\frac{\Delta x^*(y)}{\Delta X^*}=\frac{\int u^*(y,t)dt}{\int V^*(t)dt}=\frac{\int u(y)V^*(t)dt}{\int V^*(t)dt}=u(y) .$$

Consequently, the fact that in our experiment the motion is manually controlled does not affect the behavior of the system. Although we experimentally measured relative displacement profiles we will discuss velocity profiles, a concept that is more common for flows.

In order to visualize the trajectories of the particles and to measure their displacements, the tips of some cylinders were painted with phosphorescent paint. After lighting the system, the marked rods glowed in the darkness as shown in Fig. 4. The phosphorescent cylinders were arranged in three lines separated by 5 cm approximately. By using a video camera their positions were recorded after each displacement of 2 cm of the bottom plate. The relative displacement profile between two successive images was measured using the image processing software "NIH-IMAGE" on a Macintosh personal computer [15].

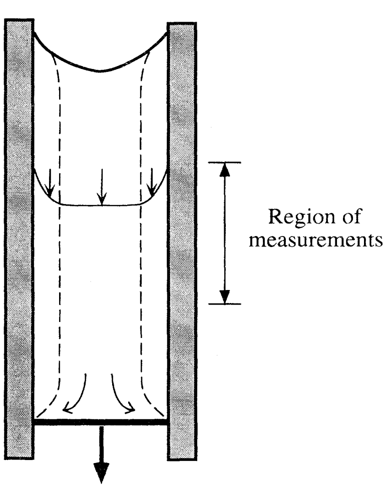

FIG. 3. Sketch of the steady flow.

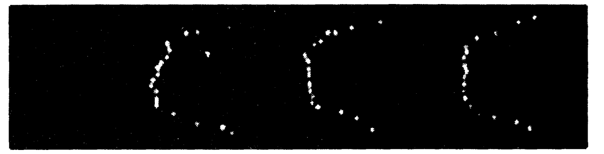

FIG. 4. Deformation of initially straight lines of marked particles.

The same image processing package was used to measure concentration profiles. A lateral picture of the channel was digitized and transformed into black and white, the threshold being chosen such that the cylinders appears in black and the void spaces in white. The concentration at a given position y^* was then defined as the ratio of the number of black pixels to the total number of pixels in a band 1.5 particle diameters wide, centered on y^*. The profile was obtained by sliding the box of measurement along y^*. This analysis was carried out at different positions x^* of the channel.

III. EXPERIMENTS IN VERTICAL BINS

Figure 5 shows relative velocity profiles for different widths of the channel $2b^*=2.5$, 4, 7, and 11 cm, corre-

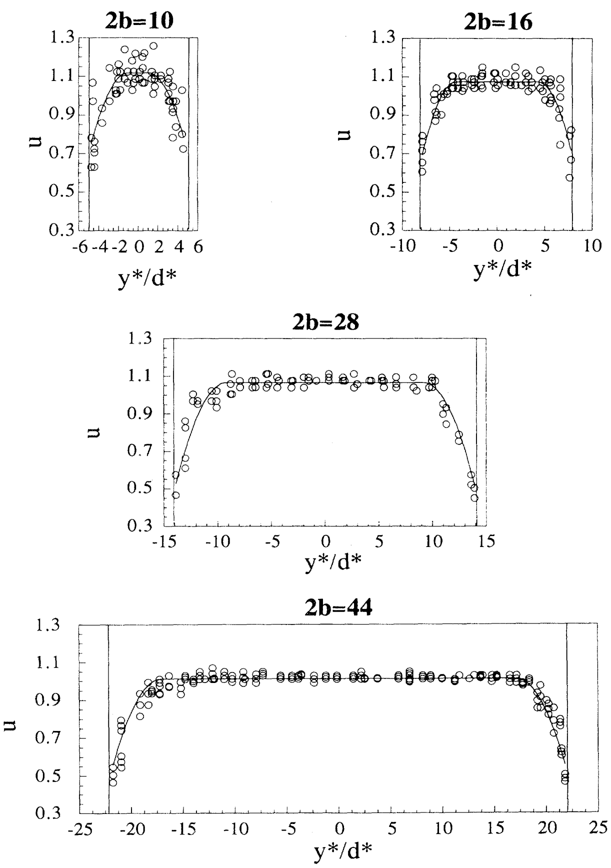

FIG. 5. Relative velocity profiles for $2b=10$, 16, 28, and 44. Continuous lines correspond to the fitting (see text).

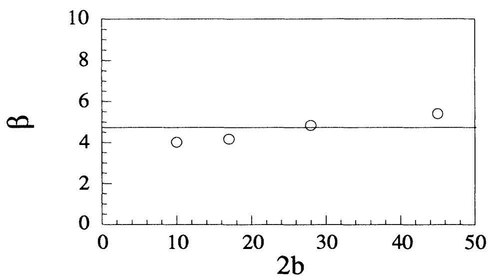

FIG. 6. Thickness β of the shear zone as a function of the bin width $2b$.

sponding to dimensionless widths in terms of particle diameters $2b=10$, 16, 28, and 44, respectively. The velocity u is plotted versus the dimensionless transversal coordinate $y=y^*/d^*$. The profiles exhibit a plug zone in the central region and two shear zones next to the walls, except for the narrowest channel $2b=10$, where the two boundary layers merge together and extend over the whole section. Note that the maximum velocity in the middle of the bin is slightly greater than 1 and increases when the width of the channel decreases. This can be explained as follows. Due to mass conservation the relative velocity has to satisfy

$$\int_{-b}^{b} u(y)dy=2b \ . \tag{1}$$

Then, as the material flows more slowly at the sidewalls, $u(y)$ exceeds 1 in the central part in order to satisfy (1).

By carefully observing the deformation of the medium during the flow, one can see that the shear zones are not the result of a continuous yield process but of intermittent local deformations. The particles do not move continuously, but experience periods without any motion punctuated by sudden jumps, usually less than 1 particle diameter. The mean profiles presented in Fig. 5 result from the integration of all these intermittent yieldings.

In order to measure the thickness of the shear zone, we fit the experimental data with a profile consisting of a central plug region and two parabolic shear zones:

$$u(y)=U_\infty-\frac{U_s}{2}(|y|-b+\beta)^2[1+\mathrm{sgn}(|y|-b+\beta)] \ ,$$

where U_∞, U_s, and β are the fitting parameters, measuring respectively the plug velocity, the slip velocity at the wall, and the thickness of the boundary layer. Figure 6 shows β against bin width. In the range of widths available in our setup, the thickness of the boundary layer is constant and equal to approximately 5 particle diameters, showing that the scale of the boundary layer is the size of the particle and not the size of the channel. Note that we did not perform experiments in very wide channels, for which Nedderman and Laohakul [5] found a slight increase in the shear zone size.

Figure 7 shows concentration profiles measured in the same channels. One clearly observes the formation of low density regions at the sidewalls where the material experiences shearing. This is related to the well-known concept of dilatancy [16], stipulating that a granular material needs to dilate in order to shear. However, it is not clear to us at this moment why the concentration boundary layer (~10 particle diameters) appears to be thicker than the velocity boundary layer (~5 particle diameters).

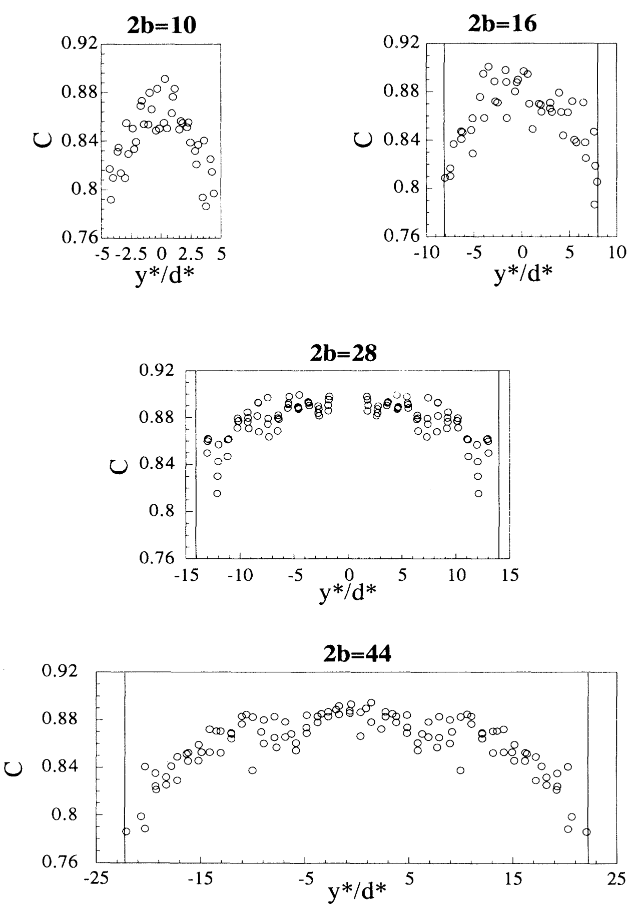

FIG. 7. Concentration profiles for $2b=10$, 16, 28, and 44.

IV. MEAN STRESSES AND FLUCTUATIONS

The experimental results have shown that the thickness of the shear zones scales on particle size and not on the bin width, which suggests that the underlying physical mechanism responsible for its formation is related to the "granularity" of the material. Consequently, any attempt to predict the velocity profile using a classical continuum model fails.

Similar questions arise for the problem of localization of failures in triaxial tests: when a granular material is compressed, the deformation is localized in shear bands whose thickness varies between 10 and 15 particle diameters [10]. A classical plastic theory can predict the phenomenon of localization by describing it as a bifurcation [17] in which the failure is treated as a discontinuity of displacement. But this approach is not able to predict the thickness of the bands. In order to explain the finite size of the failure, a Cosserat-type theory has been proposed by Mühlhaus and Vardoulakis [11] which consists in regularizing the bifurcation problem by introducing a new degree of freedom, namely, the rotation of the particles. The idea consists in adding to the description a rotation field whose variations take place on a new length scale, i.e., the particle diameter. The regularization of the equations naturally removes the discontinuity and leads to a prediction of the finite width of the failure in agreement with the experimental measurements [11].

Even though the problem of the granular vertical chute flow is strongly different from the shear bands problem (there is no instability, the flow is fully developed and stationary) one might wonder if the rotation of the particles could explain the formation of the shear zones. We thus performed numerical simulations of the vertical chute flow (a description of the molecular dynamics code we used is given later is the paper) in which the rotation of the particles was artificially blocked. The result exhibits velocity profiles similar to those obtained when the particles were free to rotate. The slip velocity at the walls was slightly greater (0.8 instead of 0.6 when the rotation is not blocked), but shear zones of 5 particle diameters were still observed. The rotation is thus not the origin of the formation of the shear zones and a Cosserat-type approach is not appropriate in explaining the velocity profiles observed experimentally.

We propose in this section an explanation of the formation of the shear zones based on the existence of stress fluctuations that are related to the granularity of the material. It is well known that in granular material the stresses are not uniformly distributed but that fluctuations exist in static configurations [18–20] as well as in rapid flows [21,22]. In this section we propose a modified Coulomb theory in which we assume that we can consider the medium as a continuum in some way, define mean stresses, and apply the equations of motion, but where we introduce stress fluctuations in addition to the mean stresses.

Let us first calculate the stress distribution of a continuum medium flowing in a vertical bin between two rough walls located at $y^*=-b^*$ and $y^*=b^*$. As we are seeking phenomena that scale on the particle size, we nondimensionalize the lengths with the particle diameter d^*, and the stresses with $\rho^* g^* d^*$, where ρ^* is the mean density of the material and g^* the gravitational acceleration. Assuming the flow to be steady and parallel to the sidewalls, the equations of motion in terms of dimensionless variables are given by

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau}{\partial y}=1 ,$$

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau}{\partial x}=0 ,$$

where τ is the dimensionless shear stress and σ_x (σ_y) the vertical (horizontal) normal stress. The x and y axes are defined in Fig. 2. The density has been assumed to be constant across the channel, because the concentration variations experimentally measured are less than 1%, and certainly have negligible effects on the stress distribution. Assuming the velocities and stresses to be uniform along the channel, the derivatives with respect to x vanish, and one obtains

$$\frac{\partial \tau}{\partial y}=1, \quad \frac{\partial \sigma_y}{\partial y}=0 . \tag{2}$$

Imposing a Coulomb-type slip condition at the walls, $|\tau|=\tan(\delta)\sigma_y$, and integrating Eqs. (2), lead to the following stress distribution:

$$\tau=y, \quad \sigma_y=\frac{b}{\tan(\delta)} . \tag{3}$$

According to the Coulomb-type criterion, a granular medium yields when along a certain plane the shear stress is equal to the normal stress times the friction coefficient. In vertical chute flow the plane of yielding is vertical at the wall. One then expects it to be close to the vertical also throughout the shear zone. Thus vertical chute flow can be seen as a system of vertical layers of granular material sandwiched between two rough walls. The formation of shear zones then results from the intermittent relative motion between the different vertical layers. According to the Coulomb criterion, this motion occurs if the shear stress $|\tau|$ reaches the yield stress $\tau_s=\tan(\phi)\sigma_y=b\tan(\phi)/\tan(\delta)$, where ϕ is the internal friction angle of the material. But as the internal friction angle of a granular material ϕ is always greater than or equal to the wall friction angle δ, no yielding should occur inside the medium, where $|\tau|=y\le\tau_s$. Assuming $\phi=\delta$, which corresponds to the experimental conditions, the threshold τ_s can only be reached at $y=\pm b$ (Fig. 8). The resulting velocity profile will be a uniform plug flow with slip at the side walls and without shear zones.

However, one can see in Fig. 8 that close to the sidewalls the value of the shear stress $|\tau|$ is not far below the yield value. Thus, if there exist fluctuations in addition to the mean stresses calculated above, one can easily imagine that the instantaneous shear stress could reach the critical value, giving rise to the yielding of the material, even away from the walls. Moreover, the difference between $|\tau|$ and τ_s is larger in the middle of the channel

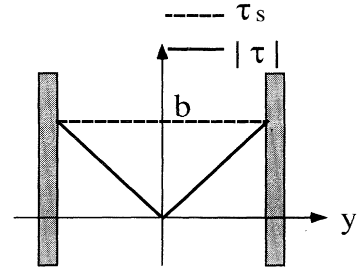

FIG. 8. Distribution of the shear and yield stress across a vertical bin.

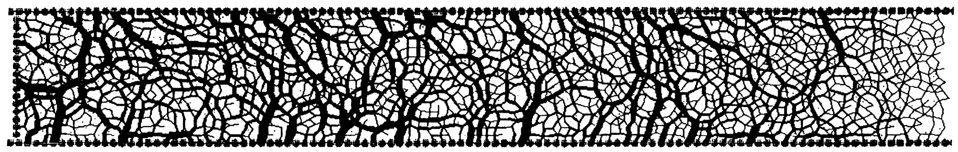

FIG. 9. Snapshot of the contact force network. The lines joining the centers of the particles in contact have a width proportional to the normal contact force.

than at the walls, which implies that the probability of yielding in the presence of fluctuations will be larger close to the walls than in the middle. Therefore the presence of stress fluctuations could explain the formation of the observed shear zones.

This explanation is based on two assumptions: the existence of stress fluctuations, and the validity of the mean stress distribution predicted by the continuum approach. As stress measurements are difficult to carry out in experiments, we performed molecular dynamics simulations of the vertical chute flow to study the stress distributions [23]. As in the experiments, the flow is created by slowly moving down a bottom plate in a channel made up of two vertical rough walls. The bin is 15 particle diameters wide and 110 particles high. In the quasistatic regime special attention has to be given to the tangential contact force. We have used the incrementally slipping model proposed by Walton and Braun [24], which is able to account for the contribution of static friction to the stress. Figure 9 is a snapshot of the contact force network during the flow. It clearly shows that stresses are not uniformly distributed over the particles and that large fluctuations exist. Looking at this figure the material can hardly be considered a continuum. However, the mean stresses obtained by averaging over 400 000 time steps are in agreement with the profiles (3) predicted by the continuum theory (Fig. 10). Normal stresses are constant across the channel, while the shear stress varies linearly with y. From the ratio τ/σ_y at the wall one gets the value of the effective wall friction $\delta=18°$. A more detailed analysis of the stress fluctuations in the simulations will be described elsewhere. For our present purpose, we conclude from these numerical simulations that the mean stress distribution predicted by the continuum theory corresponds to that measured in the simulation even when the channel is only 15 particle diameters wide, and that large spatial and temporal fluctuations exist.

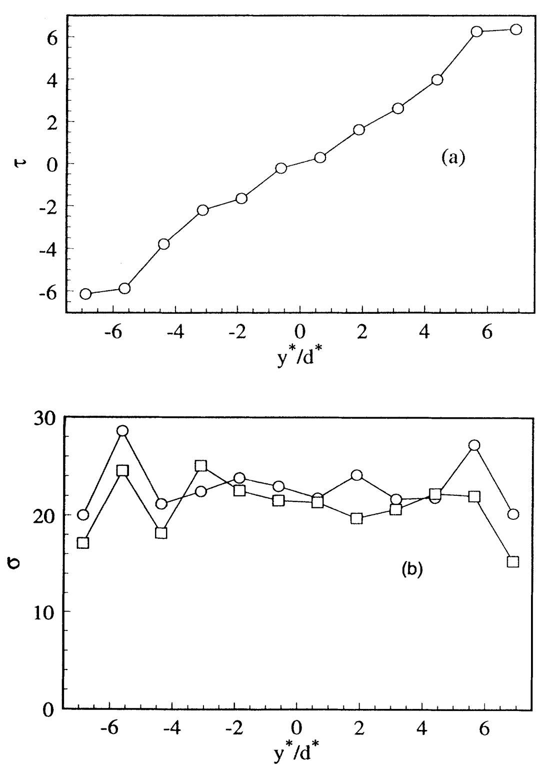

FIG. 10. Profiles of the mean shear and normal dimensionless stresses from the numerical calculations: (a) shear stress τ; (b) normal stresses σ_x(□) and σ_y(○).

V. PROBABILISTIC MODEL

In order to describe more precisely the role of the stress fluctuations as the origin of the shear zones and to predict velocity profiles, we propose a model inspired by Eyring's theory of rate processes [25]. This theory was successfully applied to explain viscosity of liquids, soil creep [26], and solid friction [27]. Its basic idea is to describe the deformation of the medium as a thermally activated process in a potential landscape. The formation of the shear zone in the vertical granular chute flow is not a thermally activated process, the thermal energy being negligible for such macroscopic particles. However, we show in this section that the fluctuations of stress can play the role of the temperature and be responsible for the yielding of the material. We propose the following model, where the probability of yielding at the location y in the channel is given by

$$p_{\text{yielding}} \sim \exp\left[-\frac{\tau_s(y)-|\tau(y)|}{\Delta\tau}\right],$$

where τ_s and τ are, respectively, the yield stress and shear stress calculated previously in Sec. IV, and $\Delta\tau$ is an empirical parameter which is a measure of the stress fluctuation magnitude. This formula simply means that the probability of yielding is greater when the difference between the shear stress and yield stress is small or when the stress fluctuations are important. In this description the stress fluctuation $\Delta\tau$ plays the role of a temperature that allows the system to overcome an energy barrier of magnitude $\tau_s - \tau$.

Therefore we propose the following expression for the velocity gradient:

$$\frac{\partial u}{\partial y} = -\operatorname{sgn}(\tau)\alpha \exp\left[-\frac{\tau_s(y) - |\tau(y)|}{\Delta\tau}\right], \tag{4}$$

where α is a constant of proportionality. The factor $\operatorname{sgn}(\tau)$ arises from the fact that the shear stress and the velocity gradient have opposite sign.

The reader has to recall that all the variables in (4) are dimensionless, implying that α and $\Delta\tau$ are dimensionless parameters. However, one can find the scaling of α and $\Delta\tau$ by writing (4) in a dimensionalized form. The parameter α which corresponds to the velocity gradient at the wall then scales with V^*/d^* where V^* is the velocity of the bottom plate and d^* the particle diameter. The magnitude of the fluctuations $\Delta\tau$ scales with $\rho^* g^* d^*$, where ρ^* is the mean density of the material and g^* the gravitational acceleration. Notice that this quantity represents the pressure created by one layer of particles. In the following the parameters α and $\Delta\tau$ are assumed to be constant across the channel, i.e., independent of y.

From (3) one obtains

$$\tau_s - |\tau| = \begin{cases} y + b & \text{for } y < 0 \\ -y + b & \text{for } y > 0 \,. \end{cases} \tag{5}$$

Substituting (5) in (4) and integrating leads to the following velocity profile:

$$u^{\pm}(y) = 1 - \alpha\Delta\tau \exp\left[\frac{\pm y - b}{\Delta\tau}\right] + \frac{\alpha\Delta\tau^2}{b}\left[1 - \exp\left[-\frac{b}{\Delta\tau}\right]\right], \tag{6}$$

where the $+$ ($-$) sign corresponds to $y > 0$ ($y < 0$). The constants of integration were found by assuming continuity at $y = 0$ and by imposing the flow rate condition (1). In order to integrate Eq. (4), we have assumed $\Delta\tau$ to be independent of y.

We do not have any experimental information about $\Delta\tau$. However, as the stress fluctuations are essentially due to the granularity of the material, one expects $\Delta\tau$ to scale on the particle size, and to be independent of the mean stresses. In our case, independence of the mean pressure implies independence of the bin width [Eqs. (3)]. So, in order to compare the theoretical predictions with the experimental results, we determine the parameters of the model α and $\Delta\tau$ by fitting one particular velocity profile with Eq. (6). The best fit for the channel of width $2b = 16$ was found with $\alpha = 0.43$ and $\Delta\tau = 0.9$. Using these values, one can predict from Eq. (6) the velocity profiles for the other channels. Figure 11 shows the agreement between the experimental measurements and the analytical predictions for different bin widths. Therefore the model is able to predict the observed velocity profiles in which the shear zone thickness does not depend on the channel width. So one can conclude that the formation of the shear zones could indeed be explained by the presence of stress fluctuations.

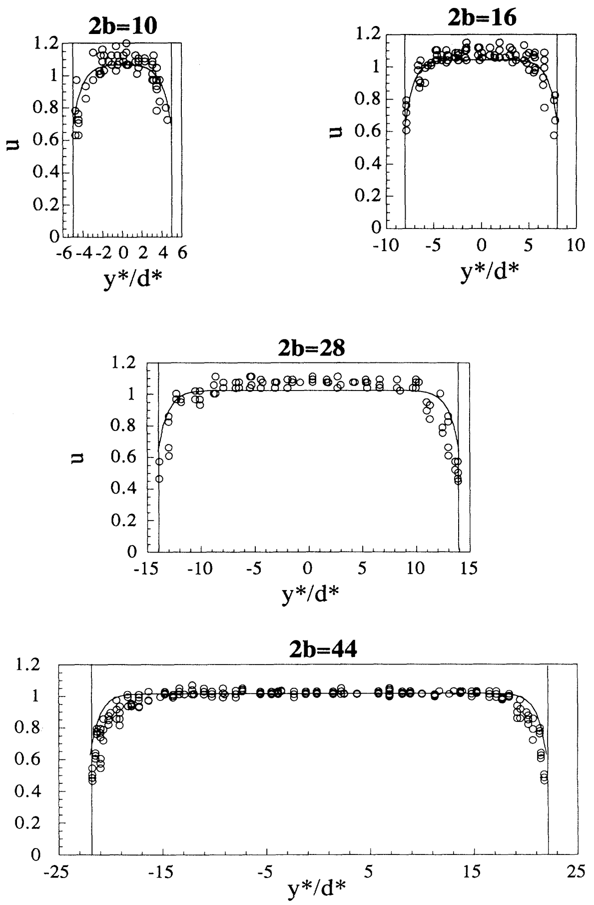

FIG. 11. Velocity profiles for $2b = 10$, 16, 28, and 44. Continuous lines: theoretical prediction for $\alpha = 0.43$ and $\Delta\tau = 0.9$. ○: experimental data.

VI. FLOWS IN INCLINED BINS

In order to further test the model we performed experiments in inclined bins. The experimental setup sketched in Fig. 2 was tilted an angle θ from the vertical (Fig. 12). The effect of the inclination has been studied for only one width of the channel, $2b = 16$. The measured velocity profiles obtained for four different angles are shown in Fig. 13. The greater the inclination, the more asymmetric the velocity profile becomes. The lower boundary layer becomes larger than the upper one and at an angle $\theta = 59.2°$ it expands across the whole channel. Thus the inclination of the bin appears to have a great influence on the flow pattern, and a strong effect on the thickness of

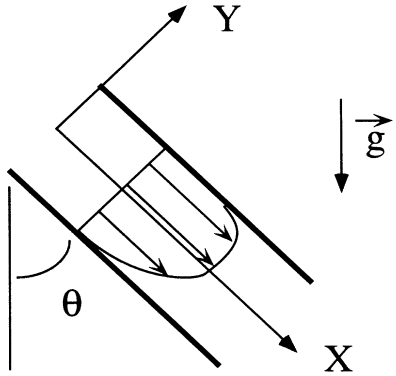

FIG. 12. Sketch of the flow in an inclined bin.

the shear zones. Note that the largest inclination where the flow is still possible is $\pi/2-\delta$.

We now show that the probabilistic model developed for the vertical flows is also able to predict this behavior. In order to do so let us first calculate the mean stress distributions in an inclined channel. Assuming the flow to be steady and uniform along the x axis (Fig. 12), the equations of motion are given by

$$\frac{\partial\tau}{\partial y}=\cos(\theta), \quad \frac{\partial\sigma_y}{\partial y}=-\sin(\theta) . \tag{7}$$

Integrating Eqs. (7) with $\delta=\phi$ and applying a Coulomb-type slide condition at the walls, the following shear stress and yield stress distributions are obtained:

$$\begin{aligned} \tau&=\cos(\theta)y-\tan(\phi)\sin(\theta)b \ , \\ \tau_s&=\cos(\theta)b-\tan(\phi)\sin(\theta)y \ . \end{aligned} \tag{8}$$

Due to a nonzero y component of the gravity, the yield stress τ_s is no longer constant across the bin section, and τ changes its sign at $y_0=\tan(\theta)\tan(\phi)b$ and not at $y=0$ (Fig. 14). One observes in Fig. 14 that $|\tau|$ and τ_s remain closer on one side of the bin than on the other one. According to the model presented in Sec. V, the energy barrier $\tau_s-|\tau|$ then grows more slowly on the left than on the right, what will give rise to an asymmetric velocity profile with boundary layers of different thicknesses.

Analytical expressions for the velocity profiles are obtained by substituting Eq. (8) in Eq. (4), and integrating:

$$u(y)=\begin{cases} \dfrac{\alpha}{S_-}\exp[S_-(y+b)]+C_1 & \text{for } y<y_0 \\ -\dfrac{\alpha}{S_+}\exp[S_+(y-b)]+C_2 & \text{for } y>y_0, \end{cases} \tag{9}$$

where

$$S_\pm=\frac{\sin(\theta)\tan(\phi)\pm\cos(\theta)}{\Delta\tau} .$$

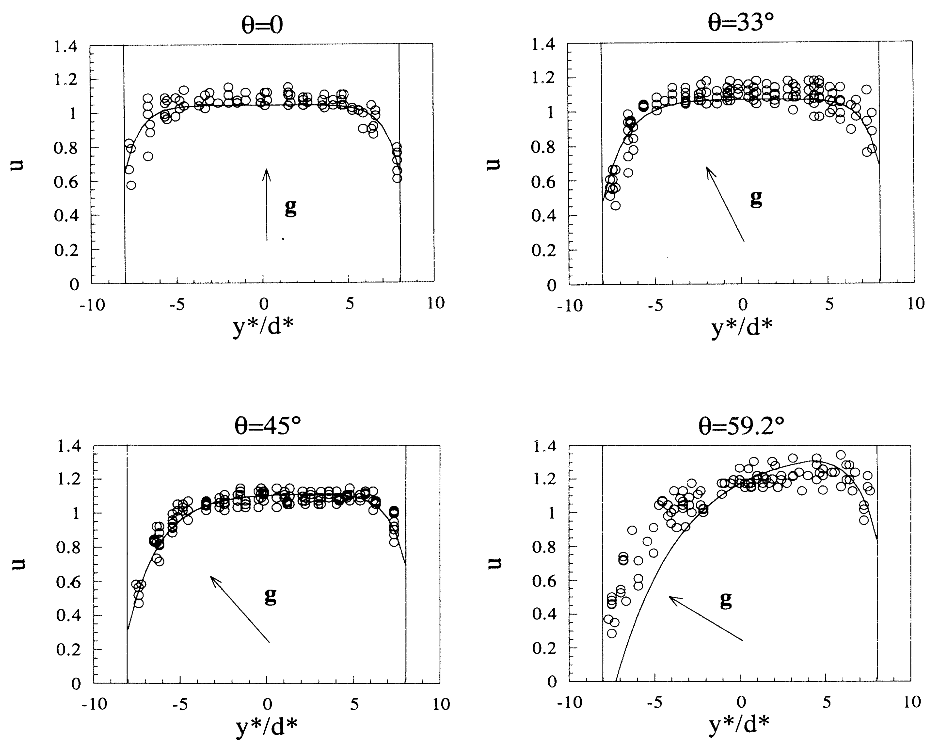

FIG. 13. Velocity profiles for $2b=16$, and $\theta=0°$, 33°, 45°, and 59.2°. Solid lines: theoretical prediction for $\alpha=0.43$ and $\Delta\tau=0.9$. ○: experimental data.

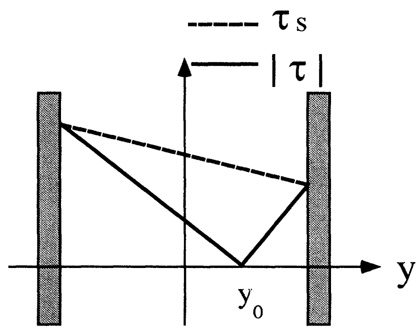

FIG. 14. Distribution of the shear and yield stress across an inclined bin.

The constants C_1 and C_2 are calculated assuming continuity at $y=y_0$ and the flow rate relation (1):

$$C_1=1+\frac{\alpha}{2b}\left[\frac{1}{S_+^2}+\frac{1}{S_-^2}\right](1-e^Z)+\frac{\alpha(S_++S_-)}{S_+(S_+-S_-)}e^Z\,, \tag{10}$$

$$C_2=1+\frac{\alpha}{2b}\left[\frac{1}{S_+^2}+\frac{1}{S_-^2}\right](1-e^Z)+\frac{\alpha(S_++S_-)}{S_-(S_+-S_-)}e^Z\,,$$

where

$$Z=S_+S_-\frac{b\Delta\tau}{\cos(\theta)}\,.$$

In order to compare the predictions of Eq. (9) with the experimental data, we chose the *same values* used in the vertical channels for the parameters of the model $\alpha=0.43$ and $\Delta\tau=0.9$. This selection is based on the assumption that the stress fluctuations do not vary with the mean stresses, which implies that $\Delta\tau$ does not depend on the bin inclination. Unlike the vertical case, the friction angle ϕ explicitly appears in the theoretical expression (9). The theoretical profiles from Eq. (9) calculated using $\phi=18°$ and the experimental results are compared in Fig. 13. The observed agreement between theory and experiments further confirms the validity of the theoretical approach presented in this work.

It has to be pointed out that in the model we assumed ϕ to be constant. However, the concentration measurements have shown that low density regions exist in the shear zones, which can have an important effect on the internal friction angle of the material. It is well known (see [28]) that the friction angle of a granular material is very sensitive to the concentration. The value of ϕ measured experimentally for the most compact piling was 21°; however, in the shear zone where the concentration decreases, ϕ is expected to be smaller. Nevertheless, the velocity profiles predicted by Eq. (9) are not very sensitive to the value of ϕ except for large inclination. Thus, if one calculates the theoretical profiles using $\phi=21°$, one finds agreement with the experimental data except for large inclination angles ($\theta=59.2°$). The best agreement between the experiments and the theory was found for $\phi=18°$, which corresponds indeed to the friction angle measured at the wall in the numerical simulations (which takes into account the dilatation due to the shear). In this case the agreement with the experimental data is good even for $\theta=59.2°$. However, for higher values of the inclination angle θ, the model predicts nonphysical velocity profiles, with regions of negative velocities near the lower wall. These discrepancies are certainly due to the approximations that have been made, namely, that the variation of concentration across the bin has been neglected, that the plane of yielding has been assumed to be vertical, and that the stress fluctuations have been described in a very simple manner. However, despite the simplicity of the analysis, the model correctly predicts the asymmetry of the velocity profile for angles less than 59°, and gives a thickness for the boundary layer which is in good agreement with the experiments.

VII. CONCLUSIONS

Quasistatic granular flows in both vertical and inclined channels have been experimentally studied, using a two-dimensional material comprised of circular cylinders. Special attention has been given to the development of boundary layers at the rough walls. In the vertical bins, the thickness of the shear zones that are created at the walls was found to be independent of the width of the bins, and equal to approximately 5 particle diameters. We have also found that the thickness of the boundary layers can easily be varied by inclining the bin. The velocity profile then becomes asymmetric, the shear zone at the lower wall being larger than at the upper one. These results cannot be explained in terms of a classical continuum theory, which is unable to predict yielding inside the material.

The interpretation we proposed for these experimental observations is based on the idea of stress fluctuations. We have shown that a simple probabilistic model is able to predict the observed velocity profiles. The basic idea is that, due to the underlying complex force network, there exist stress fluctuations that allow the material to yield even if the mean shear stress is less than the yield stress. In this model, the stress fluctuations play the role of temperature, and the difference between the shear stress and the yield stress plays the role of an energy barrier. The success of the model in predicting the experimental velocity profiles shows the relevance of the concept of stress fluctuations in explaining the formation of the shear zones.

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FIG. 4. Deformation of initially straight lines of marked particles.